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Corner Transfer Matrices of the Triangular Ising Model

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Recently, a new technique for investigating the zero-field, eight-vertex model on the square lattice using "corner transfer matrices" was suggested by Baxter. In this paper these ideas are applied to the anisotropic, ferromagnetic, triangular Ising lattice in zero field below its critical temperature. The diagonal form of the corner transfer matrix for the triangular lattice shows essentially the same structure as that for the square Ising lattice. The spontaneous magnetization can be obtained easily and agrees with that previously derived.

KEY WORDS: Ising model; triangular lattice; corner transfer matrices; spontaneous magnetization.

1. INTRODUCTION

One of the most famous models in statistical mechanics is the Ising model for ferromagnetism. Since Onsager gave the exact solution of the two-dimensional Ising model for square lattice in 1944,[1] intensive studies have been carried out on the model and various different approaches to the problem have been developed. In the 1950s several independent results for some thermodynamic properties of the triangular Ising lattice appeared simultaneously,[2–4] and various aspects of the triangular lattice have been studied thoroughly.[5]

Most recently, Baxter has initiated a new technique for the eight-vertex model on the square lattice using "corner transfer matrices."[6,7] It is the purpose of this paper to apply these ideas to the anisotropic, ferromagnetic, triangular Ising lattice below its critical temperature. It is found that the diagonal form of the corner transfer matrix for the triangular lattice shows essentially the same structure as that for the square Ising lattice proposed

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by Baxter[6,7] and that the method arrives at the same result for the spontaneous magnetization as that obtained by other approaches.[4]

We will define the corner transfer matrix (CTM) for the triangular Ising lattice in zero field in Section 2. As for the square lattice, we can diagonalize the CTM and find its eigenvalues by using spinor representations first developed by Kaufmann.[8] The essential procedure in the derivation and the results of our analysis on the representatives of the CTM are presented in Section 3. In Section 4, we conclude by giving the diagonal form of the CTM and obtain the expression for the spontaneous magnetization of the ferromagnetic triangular Ising model.

Although no new thermodynamic results have yet been obtained by this approach, it is hoped that the simplicity of the final expression for the diagonal form of the corner transfer matrix may provide illuminating insights into this model.

2. THE CORNER TRANSFER MATRIX

As the starting point of our formulation, we define the corner transfer matrix for a general anisotropic triangular Ising lattice in zero magnetic field.

Consider a hexagonal-shaped lattice plane with $2n + 1$ spins sites along the major diagonals of the hexagonal, as shown in Fig. 1. We impose the condition that the boundary spins of the lattice are all $+1$. Label the three principal directions of the lattice as **a**, **b**, and **c** and assume only nearest neighbor interactions with energies J_1, J_2, and J_3 along these three directions

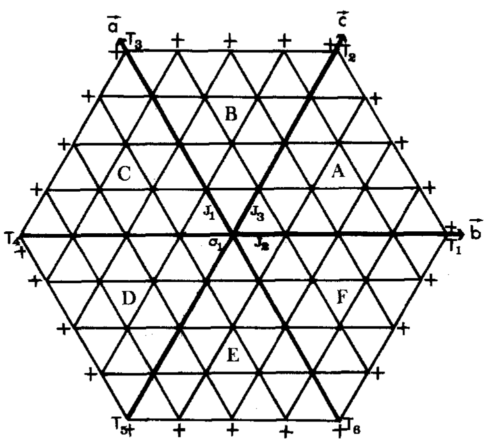

Fig. 1. A triangular lattice with hexagonal boundary and $2n + 1$ ($n = 4$) spins along each diagonal.

a, **b**, and **c**, respectively. Let $\sigma_i = \pm 1$ be the spin of the ith site; then the Hamiltonian of the system is given by

$$H = -\sum_{\text{n.n.}} J_{ij}\sigma_i\sigma_j \tag{1}$$

where the summation is over all nearest neighbors and $J_{ij} = J_1$, J_2, or J_3, depending on whether the bond between the spins σ_i and σ_j is in the direction **a**, **b**, or **c**, respectively. The partition function is given by

$$Z_n = \sum \exp(-\beta H) \tag{2}$$

where $\beta = 1/k_{\rm B}T$ and the summation here is over all possible spin configurations of the lattice.

In a similar way to Ref. 6, we divide the lattice into six "corners" (A to F) with three cuts along the major diagonals of the hexagonal. The partition is such that each corner contains all the bonds between the spins inside the corner but only bonds on the left cut, as seen from the center spin of the lattice (see Fig. 2 for corner A).

Let the center spin of the lattice be σ_1 and denote the six half-cuts of the lattice by $\tau_1,\ldots,\tau_6$ as in Fig. 1. Clearly, if all the spins on $\tau_1,\ldots,\tau_6$ are held fixed, the summation in (2) can be factorized into six parts and Z_n can be written as

$$Z_n = \sum_{\tau_1}\cdots\sum_{\tau_6} A(\tau_1\,|\,\tau_2)B(\tau_2\,|\,\tau_3)\cdots F(\tau_6\,|\,\tau_1) \tag{3}$$

where $A(\tau_1\,|\,\tau_2),\ldots, F(\tau_6\,|\,\tau_1)$ are the corner transfer matrices that account

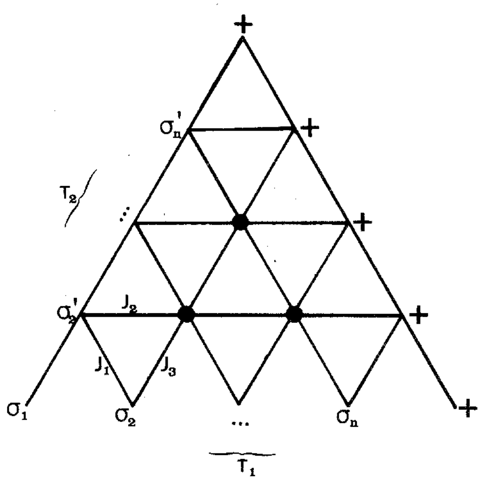

Fig. 2. Corner A of the lattice corresponding to CTM A. The summation in (4) is over all dotted spins.

for the contributions to Z_n from bonds in the respective corners. More explicitly, if we denote the spins of τ_1 by $\sigma_1, \sigma_2, ..., \sigma_n$ and those of τ_2 by $\sigma_1', \sigma_2', ..., \sigma_n'$, we have, for example, that the matrix A is given by

$$A(\tau_1 \mid \tau_2) = \delta_{\sigma_1,\sigma_1'} \sum \exp(-\beta H_{\mu\gamma}) \tag{4}$$

where $H_{\mu\gamma}$ is the interaction Hamiltonian involving all the bonds of the corner and the summation is over all interior spins of the corner as shown in Fig. 2.

Furthermore, it can be seen readily that the matrix D is the same as A, E as B, and F as C. Hence (3) can be written as

$$Z_n = \mathrm{Tr}(ABC)^2 \tag{5}$$

We further note that B is obtained from A by replacing J_1 by J_2, J_2 by J_3, and J_3 by J_1. Similarly, C is obtained from A by replacing J_1 by J_3, J_3 by J_2, and J_2 by J_1.

In the thermodynamic limit, the partition function per site is given by

$$Z = \lim_{n\to\infty} Z_n^{1/N} \tag{6}$$

where N is the total number of sites in the lattice and the free energy per site f is

$$-\beta f = \lim_{n\to\infty} \frac{1}{N} \ln Z_n \tag{7}$$

Also, the spontaneous magnetization M is just

$$M = \langle \sigma_1 \rangle = \frac{\mathrm{Tr}\{S(ABC)^2\}}{\mathrm{Tr}\{(ABC)^2\}} \tag{8}$$

where

$$S(\sigma_1, ..., \sigma_n | \sigma_1', ..., \sigma_n') = \sigma_1 \, \delta_{\sigma_1,\sigma_1'} \, \delta_{\sigma_2,\sigma_2'} \cdots \delta_{\sigma_n,\sigma_n'} \tag{9}$$

is the diagonal center-spin operator. Since the matrices A, B, and C all break into two diagonal blocks corresponding to $\sigma_1 = +1$ or -1, they commute with the diagonal matrix S.

The problem of evaluating the partition function and spontaneous magnetization therefore reduces to evaluating the eigenvalues of the product ABC of the corner transfer matrices.

3. THE REPRESENTATIVES OF THE CTM

The problem now is to diagonalize ABC. Since the method of calculation is essentially the same as the case of a square lattice,[7] we will outline the important steps here and present the results afterwards.

3.1. The Representatives of CTM A, B, and C

Define a set of anticommuting operators by

$$\begin{aligned} \Gamma_1 &= d_1, & \Gamma_2 &= c_1 s_2 \\ \Gamma_{2j-1} &= c_1 \cdots c_{j-1}\, d_j, & \Gamma_{2j} &= c_1 \cdots c_j s_{j+1}, \qquad j = 1,\dots, n \end{aligned} \tag{10}$$

where s_j, c_j, and d_j are the Pauli spin operators acting on the jth spin and are given by

$$s_j = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad c_j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad d_j = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{11}$$

$$j = 1,\dots, n$$

and

$$s_{n+1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is the identity matrix.

The set of all nonsingular $2^n \times 2^n$ matrices X such that

$$X\Gamma_l X^{-1} = \sum_{k=1}^{2n} \hat{\chi}_{kl}\Gamma_k, \qquad l = 1,\dots, 2n \tag{12}$$

for some $\hat{\chi}_{kl}$ forms a group $\mathscr{G}$. The $2n \times 2n$ matrix with elements $\hat{\chi}_{kl}$ is called the representative of X and is denoted by $\hat{X}$. The representatives have the following properties:

(i) From the anticommuting properties of the Γ_k, each representative of the group is orthogonal. Furthermore, each representative determines its parent matrix to within a multiplicative constant.

(ii) If $X_1, X_2,\dots, X_n, Y$ are members of the group such that $X_1 X_2 \cdots X_n = Y$, then the corresponding representatives $\hat{X}_1, \hat{X}_2,\dots, \hat{X}_n, \hat{Y}$ satisfy the same relationship, i.e., $\hat{X}_1\hat{X}_2\cdots\hat{X}_n = \hat{Y}$.

As for the CTMs, one can decompose them into row matrices in a similar way as for the square lattice.[6]

We consider a more symmetric corner, which includes the bonds on both of the cuts, and denote it by A' for corner A and similarly B' and C' for corners B and C, respectively. Obviously, the new corner transfer matrix A' is related to CTM A through

$$A = L_A A' \tag{13}$$

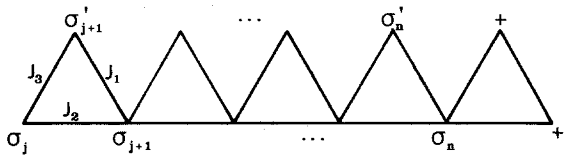

Fig. 3. The jth row of the corner A', which corresponds to the matrix G_j.

where L_A is the matrix that cancels the effect of the added bonds, or, more explicitly,

$$L_A(\sigma_1, \sigma_2,\dots, \sigma_n|\sigma_1', \sigma_2',\dots, \sigma_n')$$
$$= \{\exp[-K_2(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \cdots + \sigma_n\sigma_{n+1})]\}\, \delta_{\sigma_1,\sigma_1'} \cdots \delta_{\sigma_n,\sigma_n'} \tag{14}$$

with $\sigma_{n+1} = 1$.

The corner A' can be built up by rows of triangles as shown in Fig. 3. If we define $2^n \times 2^n$ matrices G_j ($j = 1,\dots, n$) by

$$G_j(\sigma_1, \sigma_2,\dots, \sigma_n|\sigma_1', \sigma_2',\dots, \sigma_n')$$
$$= \delta_{\sigma_1,\sigma_1'} \cdots \delta_{\sigma_j,\sigma_j'} \exp\left(\beta \sum_{k=j}^{n} J_1\sigma_{k+1}\sigma_{k+1}' + J_2\sigma_k\sigma_{k+1} + J_3\sigma_k'\sigma_{k+1}'\right) \tag{15}$$

with $\sigma_{n+1} = \sigma_{n+1}' = 1$, it can be easily seen that

$$A' = G_1G_2\cdots G_n \tag{16}$$

Furthermore, each row can be regarded as building up by triangles of spins one at a time (see Fig. 4). So each G_j can be written as

$$G_j = V_nV_{n-1}\cdots V_j \tag{17}$$

where V_j is given by

$$V_j(\sigma_1, \sigma_2,\dots, \sigma_n|\sigma_1', \sigma_2',\dots, \sigma_n')$$
$$= \delta_{\sigma_1,\sigma_1'}\, \delta_{\sigma_2,\sigma_2'} \cdots \delta_{\sigma_j,\sigma_j'}\, \delta_{\sigma_{j+2},\sigma_{j+2}'} \cdots \delta_{\sigma_n,\sigma_n'}$$
$$\times \exp[\beta(J_1\sigma_{j+1}\sigma_{j+1}' + J_2\sigma_j\sigma_{j+1} + J_3\sigma_j'\sigma_{j+1}')] \tag{18}$$

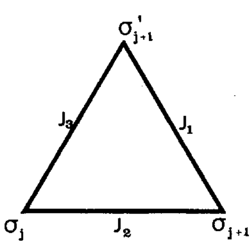

Fig. 4. The spin interactions correspond to matrix V_j.

The V_j can be written as a product of the Pauli spin operators defined in (11), i.e.,

$$\begin{aligned} V_j &= (2 \sinh 2K_1)^{1/2} \exp(K_2 s_j s_{j+1}) \\ &\qquad \times \exp(K_1{}^* c_{j+1}) \exp(K_3 s_j s_{j+1}), \qquad j = 1, 2, \ldots, n-1 \qquad (19) \\ V_n &= \exp(K_2 s_n) \exp(K_3 s_n) \end{aligned}$$

where

$$K_i = \beta J_i, \qquad \tanh K_i^* = \exp(-2K_i), \qquad i = 1, 2, 3 \tag{20}$$

Similarly, L_A can be written as

$$L_A = \prod_{j=1}^{n} \exp(-K_2 s_j s_{j+1}), \qquad s_{n+1} = 1 \tag{21}$$

It can be verified that all the matrices V_j and L_A belong to the group $\mathscr{G}$ mentioned above, e.g.,

$$\hat{V}_j = \begin{array}{r} \\ \\ \\ \\ \\ 2j-1 \\ 2j \\ 2j+1 \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccccccc} \quad & \quad & \quad & \quad & 2j-1 & 2j & 2j+1 & \quad & \quad \end{array} \\ \begin{pmatrix} 1 &&&&&&&& \\ & 1 &&&&&&& \\ && 1 &&&&&& \\ &&& \ddots &&&&& \\ &&&& \alpha\alpha' + \beta\beta'\gamma & i(\alpha\beta' + \alpha'\beta\gamma) & -\beta\delta && \\ &&&& -i(\alpha'\beta + \alpha\beta'\gamma) & \beta\beta' + \alpha\alpha'\gamma & i\alpha\delta && \\ &&&& -\delta\beta' & -i\alpha'\delta & \gamma && \\ &&&&&&& 1 & \\ &&&&&&&& \ddots \end{pmatrix} \end{array} \tag{22}$$

where

$$\begin{aligned} \gamma &= \coth 2K_1, & \delta &= \operatorname{cosech} 2K_1 \\ \alpha &= \cosh 2K_2, & \beta &= \sinh 2K_2 \\ \alpha' &= \cosh 2K_3, & \beta' &= \sinh 2K_3 \end{aligned} \tag{23}$$

and

$$\hat{L}_A = \begin{pmatrix} \alpha & -i\beta & 0 & 0 & \\ i\beta & \alpha & 0 & 0 & \\ 0 & 0 & \alpha & -i\beta & \\ 0 & 0 & i\beta & \alpha & \\ & & & & \ddots \end{pmatrix} \tag{24}$$

Hence the CTMs A, B, and C are also members of the group (i.e., their representatives under Γ_k exist).

3.2. Diagonalization of $\hat{A}\hat{B}\hat{C}$

Before we continue our analysis, we observe that the CTMs A, B, and C (hence their representatives) do not necessarily commute with each other for arbitrary J_1, J_2, and J_3. However, our aim is to diagonalize the matrix ABC $(=\mathscr{D})$, i.e., to look for a matrix, say P_2, such that

$$P_2^{-1}\mathscr{D}P_2 = \mathscr{D}_d \tag{25}$$

where $\mathscr{D}_d$ is a diagonal matrix.

As for the representative of ABC, we consider a matrix (say $\hat{P}_2$) that takes $\hat{A}\hat{B}\hat{C}$ $(=\hat{\mathscr{D}})$ to a block diagonal matrix $\hat{\mathscr{D}}_d$ given by

$$\hat{\mathscr{D}}_d = \hat{P}_2^{-1}\hat{\mathscr{D}}\hat{P}_2 = \begin{pmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \ddots & & \\ & & & \lambda_j & \\ & & & & \ddots \end{pmatrix} \tag{26}$$

where λ_j are 2×2 orthogonal matrices. Since $\hat{\mathscr{D}}$ and $\hat{\mathscr{D}}_d$ are orthogonal matrices, one can choose $\hat{P}_2$ to be orthogonal also. As $\hat{A}$, $\hat{B}$, and $\hat{C}$ are all orthogonal matrices, we may define orthogonal matrices $\hat{P}_1$ and $\hat{P}_3$ such that

$$\hat{P}_2^{-1}\hat{A}\hat{P}_3 = \hat{A}_d \tag{27a}$$

$$\hat{P}_3^{-1}\hat{B}\hat{P}_1 = \hat{B}_d \tag{27b}$$

$$\hat{P}_1^{-1}\hat{C}\hat{P}_2 = \hat{C}_d \tag{27c}$$

and require that $\hat{A}_d$, $\hat{B}_d$, and $\hat{C}_d$ are all orthogonal and of block diagonal form. For example, $\hat{A}_d$ is given by

$$\hat{A}_d = \begin{pmatrix} \lambda_1^{(1)} & & & & \\ & \lambda_2^{(1)} & & & \\ & & \ddots & & \\ & & & \lambda_j^{(1)} & \\ & & & & \ddots \end{pmatrix} \tag{28}$$

where $\lambda_j^{(1)}$ are 2×2 orthogonal matrices, and similarly for $\hat{B}_d$ and $\hat{C}_d$ with $\lambda_j^{(1)}$ replaced by $\lambda_j^{(2)}$ and $\lambda_j^{(3)}$, respectively. Obviously, we then have

$$\hat{A}_d\hat{B}_d\hat{C}_d = \hat{\mathscr{D}}_d \tag{29}$$

Note that from the set of equations (27a)–(27c) $\hat{P}_1$ is a matrix that diagonalizes the matrix $\hat{C}\hat{A}\hat{B}$ and $\hat{P}_3$ a matrix that diagonalizes $\hat{B}\hat{C}\hat{A}$ to block diagonal form.

From properties (i) and (ii) of the representatives, the corresponding matrices in $\mathscr{G}$ of the representatives $\hat{A}_d$, $\hat{B}_d$, $\hat{C}_d$, and $\hat{P}_i$ can be chosen to satisfy the set of equations (27a)–(27c) also, i.e., we can find $2^n \times 2^n$ matrices A_d, B_d, C_d, and P_i, $i = 1, 2, 3$, in $\mathscr{G}$ satisfying

$$P_2^{-1}AP_3 = A_d; \qquad P_3^{-1}BP_1 = B_d; \qquad P_1^{-1}CP_2 = C_d \tag{30}$$

Also,

$$A_dB_dC_d = \mathscr{D}_d \tag{31}$$

Since each λ_j is a 2×2 orthogonal matrix, it can be written as

$$\lambda_j = \frac{1}{2}\begin{pmatrix} \rho_j + \rho_j^{-1} & i(\rho_j^{-1} - \rho_j) \\ -i(\rho_j^{-1} - \rho_j) & \rho_j + \rho_j^{-1} \end{pmatrix} \tag{32}$$

where ρ_j is a scalar.

Consider the following $2^n \times 2^n$ matrix:

$$\prod_{j=1}^{n} \tfrac{1}{2}[(1 + \rho_j) + (1 - \rho_j)s_js_{j+1}] \tag{33}$$

One can easily show that it has the same representative $\hat{\mathscr{D}}_d$ as given by (26) and (32). From property (i) of the representatives, we conclude that it is therefore the same as $\mathscr{D}_d$ to within a multiplicative constant. Hence, if we can solve (26) or equivalently (27a)–(27c), we shall have obtained the diagonal form of $\mathscr{D}$.

We now perform the analysis on the representatives. It is convenient to group the elements of the matrices $\hat{A}$, $\hat{B}$, $\hat{C}$, and $\hat{P}_l$ ($l = 1, 2, 3$) into 2×2 blocks and write

$$\hat{A} = (a_{ij}); \qquad \hat{B} = (b_{ij}); \qquad \hat{C} = (c_{ij}) \tag{34}$$

and

$$\hat{P}_l = (p_{ij}^{(l)}), \qquad l = 1, 2, 3 \tag{35}$$

where a_{ij}, b_{ij}, c_{ij}, and $p_{ij}^{(l)}$ are all 2×2 matrices. In the limit of n large, each element tends to a limit. These limiting values are conveniently expressed in terms of generating functions defined by

$$\mathscr{A}(y, z) = \sum_{i,j} a_{ij} y^{i-1} z^{j-1} \tag{36a}$$

$$\mathscr{B}(z, x) = \sum_{i,j} b_{ij} z^{i-1} x^{j-1} \tag{36b}$$

$$\mathscr{C}(x, y) = \sum_{i,j} c_{ij} x^{i-1} y^{j-1} \tag{36c}$$

and

$$p_j^{(l)}(x) = \sum_{k=1}^{\infty} x^{k-1} p_{kj}^{(l)}, \qquad l = 1, 2, 3, \quad j = 1, 2, \ldots \tag{37}$$

From (37) we have

$$p_{kj}^{(l)} = \frac{1}{2\pi i} \int_{c_l} y^{-k} p_j^{(l)}(y)\, dy, \qquad l = 1, 2, 3 \tag{38}$$

where the counter of integration c_l is a simple closed curve surrounding the origin in the y plane within and on which $p_j^{(l)}(y)$ is analytic.

In view of Eqs. (34) and (35), the set of equations (27a)–(27c) can be expressed in the form

$$\sum_{k=1}^{\infty} a_{ik} p_{kj}^{(3)} = p_{ij}^{(2)} \lambda_j^{(1)} \tag{39a}$$

$$\sum_{k=1}^{\infty} b_{ik} p_{kj}^{(1)} = p_{ij}^{(3)} \lambda_j^{(2)} \tag{39b}$$

and

$$\sum_{k=1}^{\infty} c_{ik} p_{kj}^{(2)} = p_{ij}^{(1)} \lambda_j^{(3)} \tag{39c}$$

Hence, from (36a)–(36c), (38), and (39a)–(39c) one can easily arrive at the coupled integral equations

$$\frac{1}{2\pi i}\int_{c_3} \mathscr{A}(y, z^{-1}) p_j^{(3)}(z)\,\frac{dz}{z} = p_j^{(2)}(y)\lambda_j^{(1)} \tag{40a}$$

$$\frac{1}{2\pi i}\int_{c_1} \mathscr{B}(z, x^{-1}) p_j^{(1)}(x)\,\frac{dx}{x} = p_j^{(3)}(z)\lambda_j^{(2)} \tag{40b}$$

and

$$\frac{1}{2\pi i}\int_{c_2} C(x, y^{-1}) p_j^{(2)}(y)\,\frac{dy}{y} = p_j^{(1)}(x)\lambda_j^{(3)} \tag{40c}$$

The generating functions $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ can be evaluated explicitly. For example, $\mathscr{A}$ is found to be

$$\mathscr{A}(y, z) = \frac{1}{\Delta}\begin{pmatrix} [\alpha'(1 + z\delta\beta) - yz\alpha\gamma] & i[\beta'(1 + z\delta\beta) + yz\beta - y\delta] \\ -i[\beta'\gamma + yz\beta\gamma - z\alpha\alpha'\delta] & [\alpha'\gamma - z\alpha(\delta\beta' + y)] \end{pmatrix} \tag{41}$$

with

$$\Delta = 1 + y\delta\beta' + z\delta\beta - 2yz(\beta\beta' + \alpha\alpha'\gamma) + y^2 z\delta\beta + yz^2\delta\beta' + y^2 z^2 \tag{42}$$

where γ, δ, α, α', β, and β' are defined in (23). Note that if we put $y = e^{i\theta}$ and $z = e^{i\phi}$, (42) becomes

$$\Delta = \frac{-2yz}{\sinh 2K_1}[\cosh 2K_1 \cosh 2K_2 \cosh 2K_3 + \sinh 2K_1 \sinh 2K_2 \sinh 2K_3$$
$$- \sinh 2K_2 \cos\theta - \sinh 2K_1 \cos(\theta + \phi) - \sinh 2K_3 \cos\phi]$$

The expression in brackets is just the integrand of the double integral for the free energy of the triangular lattice.[2]

3.3. Elliptic Function Parametrization

To solve the coupled integral equations (40a)–(40c) we apply the elliptic function parametrization, which occurs naturally for the triangular lattice as follows:

$$\cosh 2K_i = \mathrm{cn}(2v_i) \tag{43a}$$

$$\sinh 2K_i = -i\,\mathrm{sn}(2v_i), \qquad i = 1, 2, 3 \tag{43b}$$

where sn, cn, and dn are Jacobian elliptic functions with modulus k given by[5]

$$k^2 = \frac{[(1 - t_1^2)(1 - t_2^2)(1 - t_3^2)]^2}{16(1 + t_1t_2t_3)(t_1 + t_2t_3)(t_2 + t_3t_1)(t_3 + t_1t_2)} \tag{44}$$

where $t_i = \tanh K_i$, $i = 1, 2, 3$.

Now if we restrict ourselves to the regime where all the K_i are real and positive (i.e., we consider only the pure ferromagnetic case), the v_i will be all purely imaginary and are subjected to the following conditions:

$$0 < \operatorname{Im} v_i < K'/2, \qquad v_1 + v_2 + v_3 = iK'/2$$

where K, K' are the complete elliptic integrals of the first kind of moduli k, $k' = (1 - k^2)^{1/2}$, respectively. (These are not to be confused with the energy coefficients K_1, K_2, and K_3 used above). If we apply the following transformations to the variables of the integral equations (40a)–(40c)

$$x = k \operatorname{sn}(u_1 + v_1) \operatorname{sn}(u_1 - v_1) \tag{45a}$$

$$y = k \operatorname{sn}(u_2 + v_2) \operatorname{sn}(u_2 - v_2) \tag{45b}$$

$$z = k \operatorname{sn}(u_3 + v_3) \operatorname{sn}(u_3 - v_3) \tag{45c}$$

we can solve the equations in terms of the new variables u_1, u_2, u_3. If we define a new kernel by

$$W_1^*(u_2, u_3)\, du_3 = \mathscr{A}(y, z^{-1})\, dz/z \tag{46a}$$

$$W_2^*(u_3, u_1)\, du_1 = \mathscr{B}(z, x^{-1})\, dx/x \tag{46b}$$

$$W_3^*(u_1, u_2)\, du_2 = \mathscr{C}(x, y^{-1})\, dy/y \tag{46c}$$

we find that each kernel can be written as a product of three matrices, i.e.,

$$W_i^*(u_j, u_l) = D^{-1}(u_j, v_j) M_i(u_j, u_l) D(u_l, v_l) \tag{47}$$

where $i\,j\,l$ are cyclic permutations of 1 2 3, and $D(u, v)$ and $M_i(u_j, u_l)$ are 2×2 matrices given by

$$D(u, v) = \begin{pmatrix} -\operatorname{cn}(u - v) & \operatorname{dn}(u + v) \operatorname{sn}(u - v) \\ \operatorname{cn}(u + v) & \operatorname{sn}(u + v) \operatorname{dn}(u - v) \end{pmatrix} \tag{48}$$

$$M_i(u_j, u_l) = \begin{pmatrix} \phi(u_l - u_j + v_j + v_l) & -\phi(u_j + u_l - v_j - v_l) \\ -\phi(u_j + u_l + v_j + v_l) & \phi(u_l - u_j - v_j - v_l) \end{pmatrix} \tag{49}$$

with

$$\phi(u) = \operatorname{dn}(u)/\operatorname{sn}(u) \tag{50}$$

Note that once more the integral equation is reduced to one involving a difference kernel form by the transformation.[9]

The coupled integral equations (40a)–(40c) become

$$\frac{1}{2\pi i} \int_{c_l'} W_i^*(u_j, u_l) p_m^{(l)}(u_l)\, du_l = p_m^{(j)}(u_j) \lambda_m^{(i)} \tag{51}$$

with $i\,j\,l$ being cyclic permutations of 1 2 3. Under the transformation (35), the contour of integration c_l' in (51) becomes a line segment $(i\eta_l - K, i\eta_l + K)$

in the u_l plane, where η_l is such that the corresponding contour c_l in the original x, y, or z (for $l = 1, 2$, or 3) plane surrounds the poles of the kernel $\mathscr{B}(z, x^{-1})$, $\mathscr{C}(x, y^{-1})$, or $\mathscr{A}(y, z^{-1})$ for $l = 1, 2$, or 3, respectively. An appropriate choice of η_l for (51) is

$$|\mathrm{Im}\, u_j| + \mathrm{Im}(v_j + v_l) < \eta_l < 2iK' - |\mathrm{Im}\, u_j| - \mathrm{Im}(v_j + v_l) \qquad (52)$$

Equations (40a)–(40c) can then be solved by defining $f_j^{(i)}(u_i) = D(u_i, v_i) \times p_j^{(i)}(u_i)$, $i = 1, 2, 3$, and expanding the functions f and ϕ in Fourier series.

3.4. The Result of the Integral Equations

On solving the equations, one obtains that the function $f_j^{(i)}(u)$ is given by

$$f_j^{(i)}(u) = \left[e^{i\pi(2j-1)u/2K} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} e^{-i\pi(2j-1)u/2K}\right] F_j^{(i)} \qquad (53)$$

where

$$F_j^{(i)} = \begin{pmatrix} -i\rho_j^{(i)} & \rho_j^{(i)} \\ i\gamma_j^{(i)} & \gamma_j^{(i)} \end{pmatrix} \qquad (54)$$

The $\rho_j^{(i)}$ and $\gamma_j^{(i)}$ are arbitrary. By the condition that the matrices P_i are orthogonal, i.e.,

$$\sum_{m=1}^{\infty} P_{mj}^{(i)T} P_{ml}^{(i)} = \delta_{jl}$$

we have a relationship between $\rho_j^{(i)}$ and $\gamma_j^{(i)}$:

$$\rho_j^{(i)} \gamma_j^{(i)} = \frac{-\pi q^{j-(1/2)}}{2kK(1 + q^{2j-1})} \qquad (55)$$

and λ_j in (15) is given by

$$\lambda_j = \lambda_j^{(1)} \lambda_j^{(2)} \lambda_j^{(3)} = \tfrac{1}{2} \begin{pmatrix} q^{j-(1/2)} + q^{(1/2)-j} & i(q^{(1/2)-j} - q^{j-(1/2)}) \\ -i(q^{(1/2)-j} - q^{j-(1/2)}) & q^{j-(1/2)} + q^{(1/2)-j} \end{pmatrix} \qquad (56)$$

where q is the "nome" of the elliptic functions given by $q = \exp(-\pi K'/K)$.

Further, the matrix $L_A^{-1/2} \mathscr{D} L_A^{1/2}$ is symmetric, so it is natural to require that $L_A^{-1/2} P_2$ be orthogonal. From the definitions (10)–(12), the representative of an orthogonal matrix X is made up of two by two diagonal subblocks. It follows that $\rho_j^{(i)}$ and $\gamma_j^{(i)}$ must be equal, so from (55)

$$\rho_j^{(i)} = \gamma_j^{(i)} = \pm i \left[\frac{\pi q^{j-(1/2)}}{2kK(1 + q^{2j-1})}\right]^{1/2} \qquad (57)$$

Also, $\hat{A}_d$, $\hat{B}_d$, and $\hat{C}_d$ take up a very simple and neat form with

$$\lambda_j^{(i)} = \frac{1}{2}\begin{pmatrix} \omega_i^{j-(1/2)} + \omega_i^{(1/2)-j} & i(\omega_i^{(1/2)-j} - \omega_i^{j-(1/2)}) \\ -i(\omega_i^{(1/2)-j} - \omega_i^{j-(1/2)}) & \omega_i^{j-(1/2)} + \omega_i^{(1/2)-j} \end{pmatrix} \tag{58}$$

where

$$\omega_i = q^{1/2} e^{-i\pi v_i/K}, \qquad i = 1, 2, 3 \tag{59}$$

Unfortunately, the matrices P_1, P_2, and P_3 do not appear to have any simple structure, but we do note from (48), (53), and (57) that P_1 depends only on k and v_1 (not on v_2 or v_3), and similarly for P_2 and P_3.

4. THE DIAGONAL FORM OF THE CORNER TRANSFER MATRICES AND THE MAGNETIZATION

Back to the original CTM, it is clear from (26), (32), (33), and (56) that the matrix $\mathscr{D}_d$ can be written

$$\mathscr{D}_d = \text{const} \times \prod_{j=1}^{\infty} \tfrac{1}{2}[(1 + q^{j-(1/2)}) + (1 - q^{j-(1/2)}) s_j s_{j+1}] \tag{60}$$

We now consider the following transformation of the representation (which corresponds to merely rearranging the rows and columns of the matrices). We replace $\sigma_1,\dots, \sigma_n$ and $\sigma_1',\dots, \sigma_n'$ by the new spin variables $\mu_1,\dots, \mu_n$ and $\mu_1',\dots, \mu_n'$ through

$$\mu_j = \sigma_j \sigma_{j+1}, \qquad \mu_j' = \sigma_j' \sigma_{j+1}', \qquad j = 1,\dots, n-1 \tag{61}$$

and $\mu_n = \sigma_n$, $\mu_n' = \sigma_n'$. Under this transformation, we have

$$s_j s_{j+1} \to s_j, \qquad s_1 \to s_1 \cdots s_n$$

with the new s_j, c_j, d_j defined under the new spin representation μ_j and μ_j'. Hence, the matrix S in (9) under the new representation is given by

$$S = s_1 s_2 \cdots s_n = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \cdots \tag{62}$$

and Eq. (60) becomes

$$\mathscr{D}_d = \text{const} \times \prod_{j=1}^{n} \tfrac{1}{2}[(1 + q^{j-(1/2)}) + (1 - q^{j-(1/2)}) s_j] \tag{63}$$

The diagonal form of the matrix $\mathscr{D}$ can therefore be written in a direct product form

$$\mathscr{D}_d = \text{const} \times \begin{pmatrix} 1 & 0 \\ 0 & q^{1/2} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & q^{3/2} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & q^{5/2} \end{pmatrix} \otimes \cdots \tag{64}$$

Actually in this representation, A_d, B_d, and C_d in Eq. (30) also take up this simple form [as seen from (28) and (58)], e.g.,

$$A_d = \text{const} \times \begin{pmatrix} 1 & 0 \\ 0 & \omega_1^{1/2} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & \omega_1^{3/2} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & \omega_1^{5/2} \end{pmatrix} \otimes \cdots \tag{65}$$

Finally, from (8), the spontaneous magnetization is given by

$$\begin{aligned} M &= \frac{\mathrm{Tr}\{S(ABC)^2\}}{\mathrm{Tr}\{(ABC)^2\}} \\ &= \frac{(1-q)(1-q^3)(1-q^5)\cdots}{(1+q)(1+q^3)(1+q^5)\cdots} \\ &= (k')^{1/4} \\ &= (1-k^2)^{1/8} \end{aligned}$$

which is just the exact result obtained by other methods.[4,5]

5. CONCLUSION

The most interesting result obtained in this paper is the diagonal form of the product of the CTMs A, B, C. It can be written in a direct product form in essentially the same way as the square lattice. It suggests that this may be an inherent characteristic of a ferromagnetic Ising model independent of the type of lattice. The matrix A_d (similarly B_d or C_d) obtained in this paper depends on v_1 (v_2 or v_3) only through $\omega_1 = \exp[-i\pi(v_1 - \frac{1}{2}iK')/K]$. Hence it can be written in the form $\exp[(v_1 - \frac{1}{2}iK')\mathscr{H}]$, where $\mathscr{H}$ is a diagonal matrix independent of v_1 (and v_2, v_3).

Although each CTM A (B or C) depends on all the three parameters v_1, v_2, and v_3, yet, as seen from (30),

$$A = P_2 A_d P_3^{-1}$$

and from the fact that P_i depends only on v_i ($i = 1$, 2, or 3), the CTMs can be factorized into a product of three matrices each of which depends on only one of the parameters in the limit of an infinite lattice. For example,

$$A = P(v_2) A_d(v_1) P(v_3)$$

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